Gröbner bases in ALG48

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1 Systems of Polynomial Equations

Assume we have a set of polynomial equations $f_i(x_1, x_2, ..., x_n) = 0$, say for example

$$x^2 + yz - 2 = 0 \tag{1}$$
$$xz + y^2 - 3 = 0 \tag{2}$$
$$xy + z^2 - 5 = 0 \tag{3}$$

One can generalize the idea of Gaussian elimination to multivariate equations by using the concept of spolynomials

$$spoly(f, g) = lcm(LT(f), LT(g))(\frac{f}{LT(f)} - \frac{g}{LT(g)})$$

where $LT(f)$ returns the leading term of the polynomial f. In lexicographic ordering, which is used throughout ALG48 ,we would thus get leading terms x^2, xz and xy for the above system ($x \geq y \geq z$). We immediately note that the spolynomial calculation cancels the leading terms from both the input polynomials, and since the spolynomial is a linear combination of the original equations it still has the same roots as the input polynomials.

For the system above we get 3 different spolynomials from the possible pairs of polynomials $(1, 2)$, $(1, 3)$ and $(2, 3)$

$$-xy^2 + 3\,x + yz^2 - 2\,z = 0$$
$$-xz^2 + 5\,x + y^2z - 2\,y = 0$$
$$y^3 - 3\,y - z^3 + 5\,z = 0$$

Next we note that the leading terms of the first two polynomials are multiples of the leading terms of some polynomials in the original system, thus we may "reduce" the spolynomials by substracting proper multiples of the original polynomials to obtain further reductions as follows

$$\left(-xy^2 + 3\,x + yz^2 - 2\,z\right) + y\left(xy + z^2 - 5\right) = 3\,x + 2\,yz^2 - 5\,y - 2\,z = 0$$
$$\left(-xz^2 + 5\,x + y^2z - 2\,y\right) + z\left(xz + y^2 - 3\right) = 5\,x + 2\,y^2z - 2\,y - 3\,z = 0$$
$$y^3 - 3\,y - z^3 + 5\,z = 0$$

We see that the first two equations are in a sense what we would expect in a multivariate triangularization process, namely if we know y and z we can solve x from either one of the first two equations, and if we know z we can solve y from the last equation. Thus what remains is finding the univariate polynomial for z – if one exists.

To reach this goal we note that this time we can reduce the original system of equations by multiplying them with suitable constants and then substracting suitable multiples of the equations above. Obviously this process will cause a cascade of reductions in all the equations we have, so we will not repeat it here in detail.

2 Gröbner bases and the Buchberger algorithm

Buchberger gives an algorithm which completes the basis of equations so that all possible spolynomials will reduce to zero with respect to the found basis, which is then called a Gröbner basis.

We should immediately note that although Gröbner bases are in a sense minimal since any leading term is cancelled if possible – and in fact most implementations cancel terms inside the polynomial too – Gröbner bases are in no way unique. In particular they depend heavily on the chosen ordering of terms (lexicographic throughout this paper), the order in which possible pairs are selected and even in which order reductions are done.

What this means in practice is that a Gröbner basis for a given set of polynomials (which may be interpreted to be equations with r.h.s = 0) may not be the "simplest" possible. However as opposed to the resultant method of solving polynomial systems Gröbner bases are minimal in the sense that they do not introduce any extraneous roots to the respective system of polynomial equations

3 Properties of Gröbner bases

We refrain from details but instead merely list some of the most improtant properties of Gröbner bases:

- A system of equations has no solutions if and only if the respective Gröbner basis includes a constant. If the basis is fully reduced - including removal of unnecessary integer multipliers - then the reduced basis will consist only of the constant 1.
- A system has finitely many solutions if and only if for each variable x_i the basis contains a polynomial whose leading term is of the form x_i^n with $n \geq 1$, e.g. the variable appears alone.
- If the Gröbner basis with respect to lexicographic ordering is sorted according to the leading terms it is often, but not always, in "triangulular" form suitable for solving via backsubstitution.

We also mention perhaps the most important property of the Buchberger algorithm – it is slow as molasses. Naturally since lexicographic ordering is most useful for solving polynomial systems it also appears to be the worst possible of the commonly used orderings with respect to execution time. Also since polynomial reductions can be interpreted as a division by multiple polynomials the Buchberger algorithm suffers from coefficient explosion common for all polynomial remainder sequence algorithms – this may occur in intermediate calculations even if the final basis would be the constant 1.

4 Gröbner basis Commands in ALG48

GBASIS calculates the Gröbner basis for a given set of polynomials with the given lexicographic ordering (missing variables are appended to the given set in alphabetical order).

GSIMP calculates the Gröbner basis as above for a given set of side relations and then reduces a given equation with respect to the side relations.

GSOLVE calculates the Gröbner basis as above for a given set of polynomial equations and then factors the set as far as possible to obtain a set of individual solutions.

4.1 *GBASIS* example

The Gröbner basis for our first example can be calculated with *GBASIS* as follows

2: { $x^2 + yz - 2$
$xz + y^2 - 3$
$xy + z^2 - 5$ }

1: $\{\ x\ y\ z\ \}$

$\Longrightarrow$

1: $\{\ 361\,x - 88\,z^7 + 872\,z^5 - 2690\,z^3 + 2375\,z$
$361\,y + 8\,z^7 + 52\,z^5 - 740\,z^3 + 1425\,z$
$8\,z^8 - 100\,z^6 + 438\,z^4 - 760\,z^2 + 361\ \}$

From the basis it is easy to see there are exactly 8 solutions to the system of equations.

4.2 *GSIMP* example

A well known example for simplification with side relations is the following problem from the Dutch Mathematics Olympiad of 1991.

Let a,b,c be real numbers such that

$$\begin{aligned} a+b+c &= 3 \\ a^2+b^2+c^2 &= 9 \\ a^3+b^3+c^3 &= 24 \end{aligned}$$

Calculate $a^4+b^4+c^4$.

Using *GSIMP* we get the result as follows

3: $a^4+b^4+c^4$

2: $\{\ a+b+c=3$
$a^2+b^2+c^2=9$
$a^3+b^3+c^3=24\ \}$

1: $\{\ \}$

$\Longrightarrow$

1: 69

One should note that had we computed the solutions for a,b and c first using either *GBASIS* or *GSOLVE* we would have ended up with an irreducible third degree polynomial (with 3 real roots), and substituting the values back would have involved considerably more work.

4.3 *GSOLVE* example

Find the extrema of $x^3y^2\,(6-x-y)$. Taking the derivatives with respect to x and y and calling *GSOLVE* we get

2: $\{\ x^2y^2\,(18-4\,x-3\,y)$
$x^3y\,(12-2\,x-3\,y)\ \}$

1: $\{\ \}$

$\Longrightarrow$

4: $\{\ y\ \}$

3: $\{\ x\ \}$

2: { $x-3$
$y-2$ }

1: 3

meaning we have solutions $x=0$ or $y=0$ or $x=3, y=2$. Calculating the basis instead and factoring it shows the extra work *GSOLVE* has done to isolate the solutions

$$\begin{gathered} x^2y\left(8\,x^2-48\,x-9\,y^2+54\,y\right) \\ x^2y^2\left(4\,x+3\,y-18\right) \\ x^2y^2\left(y-6\right)\left(y-2\right) \end{gathered}$$

Clearly the minimum possible system with the same solutions would have been

$$\begin{gathered} xy\left(x-3\right) \\ xy\left(y-2\right) \end{gathered}$$

Obviously we could remove the powers of any factor from the basis *GBASIS FCTR* returned since we are only interested in the roots. Thus we can recalculate the basis for the following polynomials

$$\begin{gathered} xy\left(8\,x^2-48\,x-9\,y^2+54\,y\right) \\ xy\left(4\,x+3\,y-18\right) \\ xy\left(y-6\right)\left(y-2\right) \end{gathered}$$

which is

$$\begin{gathered} xy\left(4\,x+3\,y-18\right) \\ xy\left(y-6\right)\left(y-2\right) \end{gathered}$$

By substituting $y=6$ into above we get $x=0$, which explains the additional reduction *GSOLVE* has done since $x=0$ is already a solution in itself. Much more complex cases can in fact occur, but *GSOLVE* can detect most of them simply by appending any two solutions it finds together and calculating the respective basis. If the basis equals either one of the two solutions, that solution is unnecessary. Otherwise the basis would be an intersection of the two solutions, possibly 1 if the solutions are completely independant.

To give an example of a nontrivial simplification in the solutions consider the following system of equations

$$\begin{aligned} xy+xz-x+z^2-2&=0 \\ xy^2+2\,xz-3\,x+y+z-1&=0 \\ y^3+y^2z+2\,yz-3\,y+2\,z^3-3\,z&=0 \end{aligned}$$

Calculating the basis with ordering $z\geq y\geq x$, factoring it, calculating the basis for each possible combination of factors and removing obvious simplifications we would obtain the solutions

$$\begin{cases} z+y-1=0 \\ y^2-2\,y-1=0 \end{cases} \begin{cases} 2\,z+x^3-6\,x^2-8\,x+1=0 \\ 2\,y-x^3+6\,x^2+8\,x-1=0 \\ x^4-6\,x^3-9\,x^2+2\,x+1=0 \end{cases}$$

$$\begin{cases} 2\,z+x^3-6\,x^2-8\,x-1=0 \\ 2\,y-x^3+6\,x^2+8\,x-1=0 \\ x^6-12\,x^5+20\,x^4+94\,x^3+76\,x^2+16\,x-7=0 \end{cases}$$

We note that the first solution has x as a parameter while the other two solutions are finite. Thus we try appending each of the other two solutions to the first solution and calculate the basis for the larger systems.

The first calculation with the 4th degree polynomial in x returns 1, a disjoint solution, but the second one with the 6th degree polynomial returns the latter solution as is. Since this clearly implies both solutions are satisfied simultaneously for the finite set of roots, the finite solution is clearly just a special case of the first solution. This is easily verified numerically by calculating the roots of the 6th degree polynomial and solving y and z from both sets of equations. Obviously the Gröbner basis method offers a much more conclusive proof that the 3rd solution is unnecessary, and in the case of parametric solutions the independance cannot be proven numerically at all.

However while *GSOLVE* does check all pairs to remove redudant solutions, the fact that not all solutions may be found to a lowest possible degree may mean that not all reducible solutions are detected. For example for the following system

$$\begin{aligned} x^2 + y + z - 3 &= 0 \\ x + y^2 + z - 3 &= 0 \\ x + y + z^2 - 3 &= 0 \end{aligned}$$

GSOLVE returns the solutions

$$\begin{cases} x - 1 = 0 \\ y - 1 = 0 \\ z - 1 = 0 \end{cases} \begin{cases} x + 3 = 0 \\ y + 3 = 0 \\ z + 3 = 0 \end{cases} \begin{cases} x + z - 1 = 0 \\ y + z - 1 = 0 \\ z^2 - 2\,z - 1 = 0 \end{cases} \begin{cases} x + y - 1 = 0 \\ y^2 - y + z - 2 = 0 \\ z^2 - 2 = 0 \end{cases}$$

In the last solution we have $z^2 - 2 = 0$, thus we may substract it from the second polynomial in the same solution. The result then factors to $(y - z)\,(z + y - 1)$ and thus we can split the last solution with new Gröbner basis calculations to

$$\begin{cases} x - z = 0 \\ y + z - 1 = 0 \\ z^2 - 2 = 0 \end{cases} \begin{cases} x + z - 1 = 0 \\ y - z = 0 \\ z^2 - 2 = 0 \end{cases}$$

Obviously such reductions cannot be easily detected (as far as I know) and *GSOLVE* makes no attempt to even do so. For a better example see SY1.

Failure examples above may no longer be valid as the algorithms are improved during development.

5 Some Polynomial Systems and their Solutions

The examples in this section have been picked from various articles and reports on the subject of solving systems of polynomial equations. Several can be found from *Journal of Symbolic Computation*, the reports by Gräbe and *Algorithms for Computer Algebra* by Geddes et al. For those interested a list of various internet resources are listed below.

- Virtual Computer Algbera Library – Gröbner Bases

  `http://www.can.nl/CA_Library/Groebner/index.html`

- Publications by the Computer Algebra Group of the University of Leipzig

  `http://www.informatik.uni-leipzig.de/~compalg/ca-fg.html`

- A Review of CAS Mathematical Capabilities

  `http://math.unm.edu/~wester/cas_review.html`

- Gröbner Bases Algorithm & The Charasteristic Sets Method

  `http://symbolicnet.mcs.kent.edu/areas/groebner/index.html`

5.1 UNIVAR

$$x^4 - 3x^3 + 6x - 4 = 0$$

Solution in 1.66s.

$$\{\, x^2 - 2 \,\{\, x - 1 \,\{\, x - 2$$

5.2 ROOTS1

$$\sqrt{x-1} + \sqrt{x-2} = \sqrt{x+3}$$

We define auxiliary variables as follows

$$\begin{cases} y_1 = \sqrt{x-1} \\ y_2 = \sqrt{x-2} \\ y_3 = \sqrt{x-3} \end{cases}$$

and thus we get a system of equations

$$\begin{aligned} y_1 + y_2 &= y_3 \\ y_1^2 &= x - 1 \\ y_2^2 &= x - 2 \\ y_3^2 &= x + 3 \end{aligned}$$

We want to eliminate the auxiliary variables, thus we use *GSOLVE* with ordering $\{\ y_1\ y_2\ y_3\ x\ \}$ and get

$$\begin{cases} 2y_1 - 3y_3x + 8y_3 \\ 2y_2 + 3y_3x - 10y_3 \\ y_3^2 - x - 3 \\ 3x^2 - 28 \end{cases}$$

Note that if we consider only positive real roots raising to power m usually introduces an m-fold ambiguity. In practice this means we simply have to check the answers we got. The numerical values of the solutions in this case are -3.0550504633 and 3.0550504633. The former obviously causes y_1^2, y_2^2 and y_3^2 to be negative, thus we discard it. The latter solution does not cause such problems and is easily checked numerically to be a solution. Thus in this case we can express the solution as the positive root of the last polynomial, and the final solution is $x = \frac{2}{3}\sqrt{21}$.

5.3 ROOTS2

$$\sqrt{x-1} + \sqrt{x-2} = \sqrt{x+y}$$
$$\sqrt{y-1} - \sqrt{y-2} = \sqrt{x-y}$$

This system is transformed into

$$\begin{aligned} x_1 + x_2 &= x_3 \\ y_1 - y_2 &= y_3 \\ x_1^2 &= x - 1 \\ x_2^2 &= x - 2 \\ x_3^2 &= x + y \\ y_1^2 &= y - 1 \\ y_2^2 &= y - 2 \\ y_3^2 &= x - y \end{aligned}$$

Eliminating the auxiliary variables with *GBASIS* using ordering $\{\ x_1\ x_1\ x_3\ y_1\ y_2\ y_3\ x\ y\ \}$ gives the basis in 21.2s. The last two polynomials in the basis give the solution

$$\begin{cases} 79x + 105y^3 - 474y^2 + 418y + 9 \\ 21y^4 - 120y^3 + 210y^2 - 108y + 1 \end{cases}$$

The polynomial in y gives 4 real roots

$$\begin{cases} .00943128534889 \\ .895009381599 \\ 2.02767870828 \\ 2.78216633906 \end{cases}$$

and the corresponding values of x from the first equation are

$$\begin{cases} -.163293714771 \\ -.996187381608 \\ 2.74572428581 \\ 2.9851853819 \end{cases}$$

Clearly the first two solutions would make x_1^2, x_2^2, y_1^2 and y_2^2 negative so we discard them. x_3^2 and y_3^2 are positive for the remaining two solutions, and inserting the answers into the original equations shows them to be true solutions. Thus the solutions are

$$\begin{cases} x = 2.74572428581 \\ y = 2.02767870828 \end{cases} \begin{cases} x = 2.9851853819 \\ y = 2.78216633906 \end{cases}$$

5.4 MINPOLY1

What is the minimal polynomial for $\sqrt{5 + 2\sqrt{6}}$?

Transforming the problem into form

$$\begin{aligned} y &= x_1 \\ x_1^2 &= 5 + 2x_2 \\ x_2^2 &= 6 \end{aligned}$$

and solving for y with ordering $\{\ x_1\ x_2\ y\ \}$ we get the basis

$$\begin{array}{r} -x_1 + y \\ 2x_2 - y^2 + 5 \\ y^4 - 10y^2 + 1 \end{array}$$

Since the last polynomial is irreducible, it is the minimal polynomial for $\sqrt{5 + 2\sqrt{6}}$.

5.5 MINPOLY2

What is the minimal polynomial for $\sqrt{5 + 2\sqrt{6}} + \sqrt{5 - 2\sqrt{6}}$?

Transforming the problem into form

$$\begin{aligned} y &= x_1 + x_2 \\ x_1^2 &= 5 + 2x_3 \\ x_2^2 &= 5 - 2x_3 \\ x_3^2 &= 6 \end{aligned}$$

and solving for y we get the polynomial $y^4 - 20y^2 + 9$, which factors into $(y^2 - 8)(y^2 - 12)$. Checking the numerical values we deduce the minimal polynomial is $y^2 - 12$, meaning the value of the original expression is actually $\sqrt{12} = 2\sqrt{3}$. The polynomial $y^2 - 8$ is in fact the minimal polynomial for $\sqrt{5 + 2\sqrt{6}} - \sqrt{5 - 2\sqrt{6}}$.

5.6 TUTO

$$
\begin{aligned}
1-x-xy^2-xz^2&=0\\
1-y-x^2y-yz^2&=0\\
1-z-zx^2-y^2z&=0
\end{aligned}
$$

Solution in 123.56s. Intermediate overflow.

$$
\begin{cases} x-z \\ y-z \\ 2z^3+z-1 \end{cases}
\begin{cases} x+y+z \\ y^2+yz+z^2+1 \\ z^3+z+1 \end{cases}
\begin{cases} 2x-2z^3+4z^2-3z-1 \\ 2y-2z^3+4z^2-3z-1 \\ 2z^4-2z^3+z^2+2z+1 \end{cases}
$$

$$
\begin{cases} x-z \\ y+2z^3+2z-1 \\ 2z^4+3z^2-z+1 \end{cases}
\begin{cases} x+2z^3+2z-1 \\ y-z \\ 2z^4+3z^2-z+1 \end{cases}
$$

5.7 ART1

$$
\begin{aligned}
x_1\,x_3-3\,x_2&=1\\
-2\,x_4\,x_1-x_3\,x_2\,x_4+2\,x_3&=0\\
x_2\,x_4^2-x_4\,x_1+x_3&=0
\end{aligned}
$$

Solution in 16.84s.

$$
\begin{cases} x_1-x_2x_4+2 \\ 2x_2{x_4}^2+3x_2-4x_4+1 \\ x_3+2x_4 \end{cases}
\begin{cases} 3x_2+1 \\ x_3 \\ x_4 \end{cases}
\begin{cases} x_1x_3-1 \\ x_1x_4-x_3 \\ x_2 \\ {x_3}^2-x_4 \end{cases}
$$

5.8 ART2

$$
\begin{aligned}
xy+2\,x^2-y+1&=0\\
2\,y^2-3\,xy-x-2&=0\\
2\,x^2-3\,y+1+2\,y^3-3\,y^2x&=0
\end{aligned}
$$

Solution in 4.99s.

$$
\begin{cases} x \\ y-1 \end{cases}
\begin{cases} 2x-6y-5 \\ 14y^2+21y+9 \end{cases}
$$

5.9 EQ1

$$
\begin{aligned}
3\,x_1+4\,x_2-2\,x_3+x_4&=-2\\
x_1-x_2+2\,x_3+2\,x_4&=7\\
4\,x_1-3\,x_2+4\,x_3-3\,x_4&=2\\
-x_1+x_2+6\,x_3-x_4&=1
\end{aligned}
$$

Solution in 3.50s.

$$
\begin{cases} 2x_1-1 \\ x_2+1 \\ 4x_3-3 \\ x_4-2 \end{cases}
$$

5.10 EQ7

$$
\begin{aligned}
cx + (c+1)\,y + z &= 1\\
x + cy + (c+1)\,z &= 2\\
(c+1)\,x + y + cz &= -1
\end{aligned}
$$

Solution in 3.33s with variables $\{\ x\ y\ z\ c\ \}$.

$$
\begin{cases}
x + cz\\
y - zc^2 + 1\\
zc^3 + z - c - 2
\end{cases}
$$

Note that if c is to be interpreted as a parameter, Maple would also solve the system for any parameter multipliers appearing in the leading terms. In this case it would append $c^3 + 1$ to the original system and then solve it. In this particular case we would get no solutions, meaning the full solution is actually

$$
\begin{cases}
x + cz\\
y - zc^2 + 1 \qquad (c^3 + 1 \neq 0)\\
\left(c^3 + 1\right) z - c - 2
\end{cases}
$$

Note that if we set $c^3 + 1 = 0$ the last polynomial in the solution would be $-c - 2$, which clearly cannot be zero simultaneously with $c^3 + 1$.

5.11 EQ8

$$
\begin{aligned}
x^2c + xy - yc - 1 &= 0\\
2\,xy^2 + yc^2 - c^2 - 2 &= 0\\
x + y^2 - 2 &= 0
\end{aligned}
$$

Solution in 101.28s with variables $\{\ x\ y\ c\ \}$. Intermediate overflow. Note that we do not interpret any variable to be a parameter in Gröbner basis calculations, thus if c is to be interpreted as a parameter the latter solution is extraneous.

$$
\begin{cases}
x - 1\\
y - 1
\end{cases}
$$

$$
\begin{cases}
19696x - 587c^8 - 742c^7 + 8056c^6 + 16070c^5 + 4998c^4 - 1962c^3 + 20200c^2 - 6892c - 21752\\
19696y + 409c^8 - 498c^7 - 5848c^6 + 2862c^5 + 10694c^4 - 3666c^3 - 26976c^2 + 49764c - 8600\\
c^9 - 14c^7 - 10c^6 + 10c^5 - 2c^4 - 56c^3 + 64c^2 - 24c - 8
\end{cases}
$$

5.12 EQ14

$$
\begin{aligned}
43 - 52\,x - 96\,y + 4\,z + 5\,yz + 26\,xz + 2\,xy &= 0\\
-69 - 35\,y - 8\,yz - 14\,xy + 3\,xz - 75\,z^2 &= 0\\
-44 - 3\,xz - 78\,xy - 8\,y^2 + 8\,z^2 &= 0
\end{aligned}
$$

Solution in 60.03s. Intermediate overflow.

$$
\begin{cases}
3.07089285252e44\,x - 3.2480628068e40\,z^6 + \ldots\\
5.13526232863e41\,y - 3.45508260975e36\,z^6 + \ldots\\
2015808044\,z^7 - \ldots
\end{cases}
$$

5.13 EX14

$$
\begin{aligned}
x^2y + 4\,y^2 - 17 &= 0 \\
2\,xy - 3\,y^3 + 8 &= 0 \\
xy^2 - 5\,xy + 1 &= 0
\end{aligned}
$$

Solution in 1.55s. No solutions.

5.14 EX16

$$
\begin{aligned}
xy + xz - x + z^2 - 2 &= 0 \\
xy^2 + 2\,xz - 3x + y + z - 1 &= 0 \\
y^3 + y^2z + 2\,yz - 3\,y + 2\,z^2 - 3\,z &= 0
\end{aligned}
$$

Solution in 8.70s.

$$
\begin{cases} y + z - 1 \\ z^2 - 2 \end{cases}
\begin{cases} x - z^2 + 2 \\ y + z \\ z^4 + 2z^3 - 5z^2 - 4z + 5 \end{cases}
$$

5.15 EX18

$$
\begin{aligned}
x^2 + yz - 2 &= 0 \\
y^2 + xz - 3 &= 0 \\
xy + z^2 - 5 &= 0
\end{aligned}
$$

Solution in 13.57s.

$$
\begin{cases} 361x - 88z^7 + 872z^5 - 2690z^3 + 2375z \\ 361y + 8z^7 + 52z^5 - 740z^3 + 1425z \\ 8z^8 - 100z^6 + 438z^4 - 760z^2 + 361 \end{cases}
$$

5.16 CFM

$$
\begin{aligned}
N &= C_n \\
A + F + 2\,B + 2\,C &= C_f \\
HF &= K_xA \\
B &= AFK_y \\
C &= K_zA^2 \\
HO &= K_w \\
H + N &= O + B + F
\end{aligned}
$$

The variables to solve are $\{\ A\ B\ C\ F\ H\ N\ O\ \}$, the problem is choosing the best ordering, my first random guess on Maple caused it to run out of memory. Given the complexity of the solutions we settle for using *GBASIS* only instead of *GSOLVE*. Testing with and without critetion 2 shows it to be quite effective for this problem.

- Order $\{\ N\ B\ C\ F\ H\ O\ A\ \}$ gives a basis of 23 polynomials in 19 minutes. Most of the polynomials have mixed leading terms containing powers of H, O and A. The size of the symbolic result on my calculator was 25304.5 bytes.
- Order $\{\ F\ N\ O\ B\ C\ H\ A\ \}$ takes 202ss to find a basis of 14 polynomials. Size of result 6709.5 bytes.

- Order { $N\ O\ B\ C\ H\ F\ A$ } takes 246s to find a basis of 15 polynomials. Size of result 7037.5 bytes.
- Order { $N\ O\ H\ B\ C\ F\ A$ } takes 258s to find a basis of 15 polynomials. Size of result 7037.5 bytes.
- Order { $N\ O\ B\ H\ C\ F\ A$ } takes 261.0s to find a basis of 15 polynomials. Size of result 7037.5 bytes.
- Order { $N\ O\ B\ C\ F\ H\ A$ } takes 194.4s to find a basis of 14 polynomials. Size of result 6548 bytes. Maple finds the basis for this ordering relatively easily. Maple GSOLVE also manages to split the solution into 20 subsolutions, most of which are concerned with special cases of the extra parameters such as $K_y = K_z = 0$, $K_y = K_z$, $K_w = K_x = 0$, $K_x = 0$, $K_w = 0$, $C_f = 0$, $K_z = 0$, $K_y + 2C_f {K_y}^2 - K_z = 0$, $K_y = K_z = 0$, $K_w = K_x = K_z = 0$, $K_x = K_z = 0$.

We note that the best orderings are such that the variables which can be solved immediately from the system are first in the ordering.

To test Maple on this problem do `with(grobner)` and try for example

```
gbasis([N-Cn,A+FF+2*B+2*C-Cf,H*FF-Kx*A,B-A*FF*Ky,C-Kz*A^2,H*Oh-Kw,H+N-Oh-B-FF],
       [N, Oh, B, C, FF, H, A, Cf, Cn, Kw, Kx, Ky, Kz], plex);
```

or

```
gsolve([N-Cn,A+FF+2*B+2*C-Cf,H*FF-Kx*A,B-A*FF*Ky,C-Kz*A^2,H*Oh-Kw,H+N-Oh-B-FF],
       [N, Oh, B, C, FF, H, A, Cf, Cn, Kw, Kx, Ky, Kz]);
```

We should also note that Maple allows a further set of identifiers as input to determine which identifiers are nonzero. This means division by that variable becomes possible, and in particular that variable becomes meaningless in testing reducibility by other polynomials since either polynomial can be scaled by the given set of variables. What this means is that we would be more likely to get a smaller solution set since the Buchberger algorithm no longer has to maintain trivial solutions in calculating a basis. This has not been implemented in ALG48, and probably never will be since one may argue Gröbner bases are already pushing the limits of HP48 calculators a bit too far – atleast until somebody invents a faster method than the Buchberger algorithm.

5.17 SY1

$$\begin{aligned} w + x + y + z &= 0 \\ wz + wx + xy + yz &= 0 \\ xwz + wyz + wxy + xyz &= 0 \\ wxyz - 1 &= 0 \end{aligned}$$

Solution in 32.36s.

$$\begin{cases} w + y \\ x + z \\ yz + 1 \end{cases} \begin{cases} w + y \\ x + z \\ yz - 1 \end{cases} \begin{cases} w + x + 2z \\ x^2 + 2xz - 1 \\ y - z \\ z^2 + 1 \end{cases}$$

The last solution is a special case of the first one. *GSOLVE* misses it because it does not know how to simplify the last solution into

$$\begin{cases} w + z \\ x + z \\ y - z \\ z^2 + 1 \end{cases}$$

which is what the merged system for solutions 1 and 3 would give. The reason is the polynomial $x^2 + 2xz - 1$, which with the side relation $z^2 + 1 = 0$ factors to $(x + z)^2$.

5.18 SY2

$$\begin{aligned} x^2+y+z-3&=0\\ x+y^2+z-3&=0\\ x+y+z^2-3&=0 \end{aligned}$$

Solution in 10.64s. Gräbe seems to have made a typing error in the solution given by Axiom.

$$\begin{cases} x+z-1\\ y+z-1\\ z^2-2z-1 \end{cases} \begin{cases} x+y-1\\ y^2-y+z-2\\ z^2-2 \end{cases} \begin{cases} x-1\\ y-1\\ z-1 \end{cases} \begin{cases} x+3\\ y+3\\ z+3 \end{cases}$$

5.19 SY3

$$\begin{aligned} x^4+x+1&=0\\ y^4+x+1&=0 \end{aligned}$$

Solution in 14.23s.

$$\begin{cases} x+y^4+1\\ y^8+2y^4+y^2+1 \end{cases} \begin{cases} x+y\\ y^4-y+1 \end{cases} \begin{cases} x-y\\ y^4+y+1 \end{cases}$$

With ordering $\{\ y\ x\ \}$ we would of course get the original system back as it is already in the final form.

5.20 SY4

$$\begin{aligned} u_0+2\,u_1+2\,u_2+2\,u_3-1&=0\\ {u_0}^2+2\,{u_1}^2+2\,{u_2}^2+2\,{u_3}^2-u_0&=0\\ 2\,u_0u_1+2\,u_1u_2+2\,u_2u_3-u_1&=0\\ 2\,u_0u_2+{u_1}^2+2\,u_1u_3-u_2&=0 \end{aligned}$$

Solution in 92.15s. Intermediate overflow.

$$\begin{cases} 3u_o-1\\ u_1\\ u_2\\ 3u_3-1 \end{cases} \begin{cases} u_0-1\\ u_1\\ u_2\\ u_3 \end{cases}$$

$$\begin{cases} 168945u_0+381533328{u_3}^5-97717752{u_3}^4-12529296{u_3}^3+5057432{u_3}^2-7598u_3-147793\\ 11263u_1+5452920{u_3}^5-1977048{u_3}^4-589356{u_3}^3+177864{u_3}^2+17866u_3-4768\\ 16895u_2-272560464{u_3}^5+78514596{u_3}^4+15104988{u_3}^3-5196676{u_3}^2-95246u_3+60944\\ 42768{u_3}^6-16848{u_3}^5-432{u_3}^4+904{u_3}^3-72{u_3}^2-12u_3+1 \end{cases}$$

5.21 SY5

$$\begin{aligned} x^3+y+z-3&=0\\ x+y^3+z-3&=0\\ x+y+z^3-3&=0 \end{aligned}$$

Solution in 37.56s.

$$\begin{cases} x-1\\ y-1\\ z-1 \end{cases} \begin{cases} x-z\\ y-z\\ z^2+z+3 \end{cases} \begin{cases} x+y+z\\ y^2+yz+z^2-1\\ z^3-z-3 \end{cases} \begin{cases} 2x+z^3-3\\ 2y+z^3-3\\ z^6-2z^4-6z^3+4z^2+6z+5 \end{cases}$$

$$\begin{cases} x-z\\ y+z^3+z-3\\ z^6+z^4-6z^3+z^2-3z+8 \end{cases} \begin{cases} x+z^3+z-3\\ y-z\\ z^6+z^4-6z^3+z^2-3z+8 \end{cases}$$

5.22 SY6

$$\begin{aligned} v+w+x+y+z &= 0 \\ vw+vz+wx+xy+yz &= 0 \\ vwx+vwz+vyz+wxy+xyz &= 0 \\ vwxy+vwxz+vwyz+vxyz+wxyz &= 0 \\ vwxyz-1 &= 0 \end{aligned}$$

Out of memory error. Intermediate overflow. According to Gräbe the system has 70 complex solutions. Maple did not give a result in 800s.

5.23 SY7

$$\begin{aligned} x^3+y^2+z-3 &= 0 \\ x+y^3+z^2-3 &= 0 \\ x^2+y+z^3-3 &= 0 \end{aligned}$$

Out of memory error. Intermediate overflow, debugging shows intermediate coefficients to be up to 150 digits long. Maple returns the solutions

$$\begin{cases} x-1 \\ y-1 \\ z-1 \end{cases} \begin{cases} x-z \\ y-z \\ z^2+2z+3 \end{cases}$$

$$
\left\{
\begin{array}{l}
2486914952008330143746938 6x+ \\
25516525693389301164344 7z^{23} - 1213617204034471920220 9z^{22}- \\
2377969790835779802246 8z^{21} - 59748368472049354019810 51z^{20}- \\
7570469185328266754374 52z^{19} + 6765997664223475013975 3z^{18}+ \\
591325763148284316891693 75z^{17} + 170473315450659317288513 90z^{16}+ \\
192038391287252406016707 6z^{15} - 306811760344137168576871 554z^{14}- \\
1521992556731207757306474 54z^{13} - 2799048030996923596877027 5z^{12}+ \\
903728873742135661065354 565z^{11} + 652665362481617082255689 001z^{10}+ \\
22740230858165506293752 9223z^{9} - 15727233154567238306627 18674z^{8}- \\
143385921835743983827998 3458z^{7} - 765062554567217444745965 468z^{6}+ \\
155975788147079832643194 8854z^{5} + 156944476990278096040536 3160z^{4}+ \\
106741048558248954972006 4256z^{3} - 618728537995078215103778 890z^{2}- \\
818981374403056838124032 335z - 459873943755090949831124 794 \\
\\
2486914952008330143746938 6y- \\
52264277739696148130953 7z^{23} - 1176195025571807078411 41z^{22}- \\
1926613897751400632163 00z^{21} + 11706606119127710215579 639z^{20}+ \\
3876177579248947796393 274z^{19} + 4228777960890861004473 829z^{18}- \\
110774734125258911957189 047z^{17} - 477021774969505547371108 10z^{16}- \\
336726333017321819765731 30z^{15} + 549779209388169342460193 280z^{14}+ \\
323346317881985679673242 846z^{13} + 148057082666808795656977 349z^{12}- \\
155387941917078799436887 0359z^{11} - 120203844611337258566922 0319z^{10}- \\
548649017641567485449454 713z^{9} + 262465515023781915037625 9634z^{8}+ \\
243698910471759125164448 1786z^{7} + 139122534352866064018031 3160z^{6}- \\
258132453002033915591627 4046z^{5} - 255872671392455007085780 6128z^{4}- \\
173772700735712052209708 3560z^{3} + 102873821012702392710067 5710z^{2}+ \\
134090074945008507793852 4225z + 675712294615812078391764 432 \\
\\
z^{24} - z^{23} - 23z^{21} + 20z^{20} + 3z^{19} + 225z^{18} - 168z^{1}7 - 66z^{1}6 - 1158z^{15}+ \\
670z^{14} + 569z^{13} + 3443z^{12} - 1357z^{11} - 2053z^{10} - 6620z^{9} + 1558z^{8} + 3548z^{7}+ \\
8678z^{6} - 1264z^{5} - 3158z^{4} - 6480z^{3} - 79z^{2} + 2042z + 1784
\end{array}
\right.
$$

5.24 SY8

$$
\begin{aligned}
ax + y - 1 &= 0 \\
x + ay - 1 &= 0
\end{aligned}
$$

Solution in 3.13s with variables $\{\ x\ y\ a\ \}$.

$$
\left\{\begin{array}{l} x - y \\ ya + y - 1 \end{array}\right.
\left\{\begin{array}{l} x + y - 1 \\ a - 1 \end{array}\right.
$$

If a is a parameter the case $a + 1 = 0$ should be checked separately (no solutions).

5.25 SY9

$$
\begin{aligned}
t - (a - v) &= 0 \\
x + y + z + t - (u + w + a) &= 0 \\
xz + xt + yz + zt - (ua + uw + wa) &= 0 \\
xzt - uwa &= 0
\end{aligned}
$$

Solution in 32.06s with variables { x y z t }.

$$\begin{cases} x+y-v-w \\ ya-yv-av+v^2+vw \\ z-u \\ t-a+v \end{cases} \begin{cases} x+y-u-v \\ ya-yv-av+uv+v^2 \\ z-w \\ t-a+v \end{cases}$$

$$\begin{cases} x+y+a-u-v-w \\ ya-yv+a^2-au-2av-aw+uv+uw+v^2+vw \\ z-a \\ t-a+v \end{cases}$$

5.26 SY10

$$\begin{aligned} d\sin\alpha_1 &= -a \\ d\cos\alpha_1 &= b \\ l_2\sin\alpha_2 + l_3\sin\alpha_3 &= c \\ l_2\cos\alpha_2 + l_3\cos\alpha_3 &= d \\ \cos{\alpha_1}^2 + \sin{\alpha_1}^2 &= 1 \\ \cos{\alpha_2}^2 + \sin{\alpha_2}^2 &= 1 \\ \cos{\alpha_3}^2 + \sin{\alpha_3}^2 &= 1 \end{aligned}$$

No solution in practical execution time with variables
{ $\sin\alpha_1$ $\sin\alpha_2$ $\sin\alpha_3$ $\cos\alpha_1$ $\cos\alpha_2$ $\cos\alpha_3$ d }

5.27 SY11

$$\begin{aligned} x^2y^2\,(-4\,x-3\,y+18) &= 0 \\ x^3y\,(-2\,x-3\,y+12) &= 0 \end{aligned}$$

Solution in 5.92s.

$$\{\, x \ \{\, y \begin{cases} x-3 \\ y-2 \end{cases}$$